Improving the precision of light quark mass determinations

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We discuss the concepts and the framework of the renormalization procedure in regularization-invariant momentum subtraction schemes. These schemes are used in the context of lattice simulations for the determination of physical quantities like light quark masses. We focus on the renormalization procedure with a symmetric subtraction point of a quark mass and discuss its conversion from the RI/SMOM scheme to the $\overline{\text{MS}}$ scheme. A symmetric subtraction point allows for a lattice calculation with a reduced contamination from infrared effects. The perturbative series for the resulting matching factor at next-to-leading order is better behaved for the symmetric momentum configuration than for the exceptional one which can decrease the uncertainty in light quark mass determinations.

I. INTRODUCTION

In continuum perturbation theory dimensional regularization [1] is for many phenomenological applications the most common regulator for divergent loop-integrals. Physical observables computed to higher orders are often conveniently renormalized in the minimal subtraction ($\overline{\text{MS}}$) scheme [2, 3]; thus the renormalization constants are well known to high order in perturbation theory.

However, the $\overline{\text{MS}}$ scheme is fixed to dimensional regularization and is usually not directly applicable in lattice calculations. In the context of lattice simulations in combination with non-perturbative renormalization, regularization-invariant momentum subtraction(MOM) schemes are used [4]. In regularization-invariant(RI) schemes the renormalization constants are often fixed by demanding that the renormalized quantity is equal to its lowest order value for a special momentum configuration(subtraction point). In order to convert results computed in a RI/MOM scheme to the $\overline{\text{MS}}$ scheme matching factors are needed which translate an observable renormalized in the given RI/MOM scheme into the $\overline{\text{MS}}$ scheme. These matching factors can be computed perturbatively. For example, the computation of the mass conversion factor $C_m^{\text{RI/MOM}}$, which turns a quark mass renormalized in the RI/MOM scheme into the $\overline{\text{MS}}$ scheme, or the conversion factor $C_q^{\text{RI/MOM}}$, which performs the corresponding conversion of the quark fields, are both known up to three-loop order in perturbative Quantum Chromodynamics(QCD) [4, 5, 6]. In the RI$'$/MOM scheme which is also used in lattice simulations these matching factors are also known to the same order [6, 7].

Apart from quark masses, also the strong coupling constant α_s has been considered in momentum subtraction schemes [8, 9, 10, 11, 12, 13]. Taking into account vertex diagrams, one has a larger choice of defining the subtraction point at which the renormalization constants are fixed through different momentum configurations. Two examples are the exceptional(or asymmetric) subtraction point with the external minkowskian momenta $p_1^2 = p_2^2 = -\mu^2$, $p_3 = 0$ or the non-exceptional(or symmetric) momentum configuration with $p_1^2 = p_2^2 = p_3^2 = -\mu^2$. The symbol μ denotes here the renormalization scale.

In Ref. [14] the renormalization of quark bilinear operators for a symmetric momentum subtraction point has been studied in the RI/SMOM scheme for the vector, axial-vector, scalar, pseudo-scalar and tensor operators, where the S in SMOM stands for symmetric. The corresponding renormalization constants Z are related among each other through the Ward-Takahashi identities. These relations have been used to perform mass renormalization with a symmetric subtraction point.

Light up-, down- and strange-quark masses were determined in Ref. [15] through lattice simulations in the RI/MOM scheme and subsequently converted to the $\overline{\text{MS}}$ scheme. However, the perturbative expansion of the matching factor $C_m^{\text{RI/MOM}}$ in QCD shows poor convergence and leads currently to a significant contribution to the error on these quark masses. The uncertainty from the renormalization procedure of the quark masses in Ref. [15] amounts to around 60% of the total error. In order to renormalize the quark masses in the lattice simulation, the renormalization constants need to be determined in a lattice calculation [16]. As shown in Ref. [16], a symmetric subtraction point involves non-exceptional momenta which imply a lattice simulation with suppressed contamination from infrared effects, whereas the asymmetric subtraction point is less suited, since effects of chiral symmetry breaking vanish only slowly for large external momenta as a result of Weinberg's theorem [17].

This proceedings contribution is organized as follows: in Section II we introduced the used conventions and notations and discuss generalities. In Section III we summarize the results for the mass renormalization with a symmetric subtraction point, addressing the matching factors and the two-loop anomalous dimensions. Finally we close with the conclusions in Section IV.

II. GENERALITIES AND NOTATION

Bare and renormalized quantities are related through the renormalization constants Z, like the renormalization constant Z_m of the quark mass m and the renormalization constant Z_q of the fermion field Ψ with the properties

$$m_R = Z_m m_B \quad \text{and} \quad \Psi_R = Z_q^{1/2}\Psi_B. \tag{1}$$

The subscripts $_R$ and $_B$ denote renormalized and bare objects. These renormalization constants can be fixed through renormalization conditions imposed on self-energy type diagrams as shown in Fig. 1 given by

$$-iS(p) = \int dx e^{ipx}\langle T[\Psi(x)\overline{\Psi}(0)]\rangle \tag{2}$$

$$= \frac{i}{\not{p} - m + i\epsilon - \Sigma(p)}, \tag{3}$$

where $\Sigma(p)$ incorporates the higher order QCD corrections.

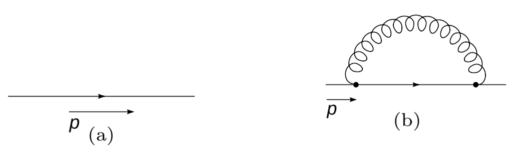

FIG. 1: QCD corrections up to one-loop order to the quark propagator.

In perturbation theory the conditions for the self-energies in the last line of Table I fix the renormalization constants in the RI/MOM scheme.

The vector and axial-vector Ward-Takahashi identities relate the self-energies and amputated Green's functions

$$\Lambda_{\hat{O}} = S^{-1}(p_2)G_{\hat{O}}S^{-1}(p_1) \tag{4}$$

of quark bilinear operators $\hat{O} = \bar{u}\Gamma_{\hat{O}}d$ with the insertion of, in general, the vector($\Gamma_V = \gamma^\mu$), axial-vector($\Gamma_A = \gamma^\mu\gamma_5$), scalar($\Gamma_S = \mathbb{1}$) or pseudo-scalar($\Gamma_P = i\gamma_5$) operators. Diagrams of these Green's functions are shown in Fig. 2.

For each of the operators a renormalization constant $Z_{\hat{O}}$ needs to be introduced with $\hat{O}_R = Z_{\hat{O}}\hat{O}_B$. In the $\overline{\text{MS}}$, RI/MOM and RI′/MOM schemes the renormalization constants are related among each other with $Z_V = 1 = Z_A$, $Z_P = 1/Z_m$ and $Z_S = Z_P$; more details can be found in e.g. Refs. [18, 19]. Some of these properties hold non-perturbatively while others are proven only in continuum perturbation theory. Through the Ward-Takahashi identities the renormalization constants in the RI/MOM scheme can also be determined

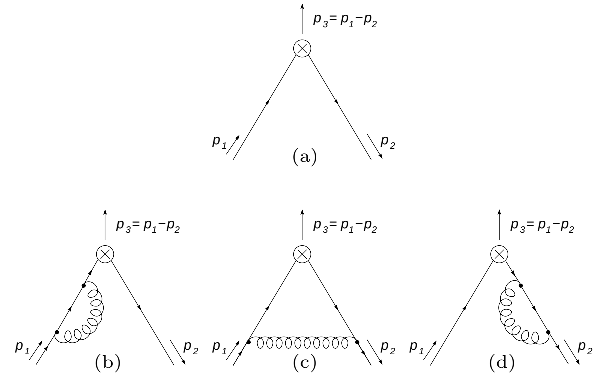

FIG. 2: Diagrams which contribute to the computation of the Green's function $G_{\hat{O}}$ at leading order (first line) and next-to-leading order(second line) in perturbative QCD. The circle with the cross denotes the inserted operator $\hat{O}$. Solid lines indicate fermions and spiral lines gluons.

by the conditions on the amputated Green's functions as given in the last row of Table I. In this case the subtraction point is exceptional, $p_1^2 = p_2^2 = -\mu^2$, $p_3 = 0$, without any momentum transfer leaving the operator. In Ref. [14] a symmetric subtraction point has been studied and its applicability demonstrated through the RI/SMOM and RI/SMOM$_{\gamma_\mu}$ schemes. The conditions for these new schemes are also summarized in Table I and the proofs for the above relations among the renormalization constants has been performed in Ref. [14]. A non-perturbative test of the RI/SMOM scheme for the quark mass renormalization can be found in Ref. [20].

Also the tensor operator, with $\Gamma_T = \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, has been studied in Refs. [7, 21, 22] and in Ref. [14] for a symmetric subtraction point.

III. RESULTS

A. Matching factors

The matching factors which convert a quark mass from the RI/SMOM or RI/SMOM$_{\gamma_\mu}$ scheme to the $\overline{\text{MS}}$ scheme

$$m_R^{\overline{\text{MS}}} = C_m^{\text{RI/SMOM}_{(\gamma_\mu)}}\, m_R^{\text{RI/SMOM}_{(\gamma_\mu)}} \tag{5}$$

have been computed to one-loop order in perturbative QCD in Ref. [14]. The numerically evaluated result allows for a direct comparison of the size of the perturbative coefficients in the different schemes. In the RI/SMOM scheme the result reads

$$C_m^{\text{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi}[0.645518856... - \xi 0.229271492...] + \mathcal{O}(\alpha_s^2). \tag{6}$$

RI/SMOM	$\lim\limits_{m_R\to 0} \frac{1}{12q^2}\mathrm{Tr}\left[q_\mu \Lambda^\mu_{V,R}(p_1,p_2)\,\not{q}\right]\Big\vert_{sym} = 1, \quad \lim\limits_{m_R\to 0} \frac{1}{12q^2}\mathrm{Tr}\left[q_\mu \Lambda^\mu_{A,R}(p_1,p_2)\gamma_5\,\not{q}\right]\Big\vert_{sym} = 1,$ $\lim\limits_{m_R\to 0} \frac{1}{12}\mathrm{Tr}\left[\Lambda_{S,R}(p_1,p_2)\mathbb{1}\right]\big\vert_{sym} = 1, \quad \lim\limits_{m_R\to 0} \frac{1}{12i}\mathrm{Tr}\left[\Lambda_{P,R}(p_1,p_2)\gamma_5\right]\big\vert_{sym} = 1,$ $\lim\limits_{m_R\to 0} \frac{1}{12p^2}\mathrm{Tr}[S_R^{-1}(p)\,\not{p}]\Big\vert_{p^2=-\mu^2} = -1,$ $\lim\limits_{m_R\to 0} \frac{1}{12m_R}\left\{\mathrm{Tr}\left[S_R^{-1}(p)\right]\big\vert_{p^2=-\mu^2} - \frac{1}{2}\,\mathrm{Tr}\left[q_\mu\Lambda^\mu_{A,R}(p_1,p_2)\gamma_5\right]\big\vert_{sym}\right\} = 1.$
RI/SMOM$_{\gamma_\mu}$	$\lim\limits_{m_R\to 0} \frac{1}{48}\mathrm{Tr}\left[\Lambda^\mu_{V,R}(p_1,p_2)\gamma_\mu\right]\big\vert_{sym} = 1, \quad \lim\limits_{m_R\to 0} \frac{1}{48}\mathrm{Tr}\left[\Lambda^\mu_{A,R}(p_1,p_2)\gamma_5\gamma_\mu\right]\big\vert_{sym} = 1,$ $\lim\limits_{m_R\to 0} \frac{1}{12}\mathrm{Tr}\left[\Lambda_{S,R}(p_1,p_2)\mathbb{1}\right]\big\vert_{sym} = 1, \quad \lim\limits_{m_R\to 0} \frac{1}{12i}\mathrm{Tr}\left[\Lambda_{P,R}(p_1,p_2)\gamma_5\right]\big\vert_{sym} = 1,$ $\lim_{m_R\to 0} \frac{1}{48}\left\{\mathrm{Tr}\left[\gamma^\mu \frac{\partial S_R^{-1}(p)}{\partial p^\mu}\right]\Big\vert_{p^2=-\mu^2} + \mathrm{Tr}\,\left[q_\mu\gamma^\alpha\frac{\partial}{\partial q^\alpha}\Lambda^\mu_{V,R}\right]\big\vert_{sym}\right\} = -1,$ $\lim_{m_R\to 0} \frac{1}{12m_R}\left\{\mathrm{Tr}\left[S_R^{-1}(p)\right]\big\vert_{p^2=-\mu^2} - \frac{1}{2}\,\mathrm{Tr}\left[q_\mu\Lambda^\mu_{A,R}(p_1,p_2)\gamma_5\right]\big\vert_{sym}\right\} = \quad 1.$
RI/MOM	$\lim\limits_{m_R\to 0} \frac{1}{48}\mathrm{Tr}\left[\Lambda^\mu_{V,R}(p_1,p_2)\gamma_\mu\right]\big\vert_{asym} = 1, \quad \lim\limits_{m_R\to 0} \frac{1}{48}\mathrm{Tr}\left[\Lambda^\mu_{A,R}(p_1,p_2)\gamma_5\gamma_\mu\right]\big\vert_{asym} = 1,$ $\lim\limits_{m_R\to 0} \frac{1}{12}\mathrm{Tr}\left[\Lambda_{S,R}(p_1,p_2)\mathbb{1}\right]\big\vert_{asym} = 1, \quad \lim\limits_{m_R\to 0} \frac{1}{12i}\mathrm{Tr}\left[\Lambda_{P,R}(p_1,p_2)\gamma_5\right]\big\vert_{asym} = 1,$ $\lim\limits_{m_R\to 0} \frac{1}{48}\mathrm{Tr}\left[\gamma^\mu \frac{\partial S_R^{-1}(p)}{\partial p^\mu}\right]\Big\vert_{p^2=-\mu^2} = -1, \quad \lim\limits_{m_R\to 0} \frac{1}{12m_R}\mathrm{Tr}[S_R^{-1}(p)]\big\vert_{p^2=-\mu^2} = 1.$

TABLE I: This table summarizes the renormalization conditions for the RI/SMOM and RI/SMOM$_{\gamma_\mu}$ schemes as defined in Ref. [14] for the amputated Green's functions of the vector, axial-vector, scalar and pseudo-scalar operators as well as for the self-energies. The last row contains the conditions for the RI/MOM scheme [4]. The subscript $(a)sym$ stands for the restriction to the (a)symmetric momentum configuration and $q = p_1 - p_2$ is the momentum transfer leaving the operator.

The corresponding conversion factor for the fermion fields $C_q^{\mathrm{RI/SMOM}} = Z_q^{\overline{\mathrm{MS}}}/Z_q^{\mathrm{RI/SMOM}}$ in the case of the RI/SMOM scheme is given by

$$C_q^{\mathrm{RI/SMOM}} = 1 - \frac{\alpha_s}{4\pi}\xi 1.333333333... + \mathcal{O}(\alpha_s^2). \quad (7)$$

In the RI/SMOM$_{\gamma_\mu}$ scheme the results for the matching factors read

$$C_m^{\mathrm{RI/SMOM}_{\gamma_\mu}} = 1 - \frac{\alpha_s}{4\pi}[1.978852189... \\ +\xi 0.666666667...] + \mathcal{O}\left(\alpha_s^2\right) (8)$$

and

$$C_q^{\mathrm{RI/SMOM}_{\gamma_\mu}} = 1 + \frac{\alpha_s}{4\pi}[1.333333333... \\ -\xi 0.437395174...] + \mathcal{O}\left(\alpha_s^2\right), (9)$$

where ξ is the gauge parameter and α_s the strong coupling constant.

In order to determine the perturbative mass conversion factors C_m for a symmetric subtraction point in the two mentioned schemes, one can consider the computation of the amputated Green's function of the (pseudo-)scalar operator or the (axial-)vector operator with suitable projectors. In the case of the pseudo-scalar and axial-vector operator a naive anti-commuting definition of γ_5 is used for the treatment of γ_5 in dimensional regularization [1, 23] which is a self-consistent prescription for the flavor non-singlet contributions [24, 25].

Comparing these matching factors with the conversion factor in the RI/MOM scheme

$$C_m^{\mathrm{RI/MOM}} = 1 - \frac{\alpha_s}{4\pi}[5.333333333... \\ +\xi 2.000000000...] + \mathcal{O}\left(\alpha_s^2\right) (10)$$

and the RI$'$/MOM scheme[26]

$$C_m^{\mathrm{RI'/MOM}} = 1 - \frac{\alpha_s}{4\pi}[5.333333333... \\ +\xi 1.333333333...] + \mathcal{O}\left(\alpha_s^2\right), (11)$$

one finds in particular in the Landau gauge ($\xi = 0$), which is usually adopted in the lattice simulations, a significantly smaller one-loop coefficient which suggests a better behaved perturbative series. If this behavior persists at higher orders it allows for a substantial reduction of the systematic uncertainty in the light quark mass determinations.

The matching factors C_m^x in the x = RI/MOM, x = RI$'$/MOM, x = RI/SMOM and x = RI/SMOM$_{\gamma_\mu}$ schemes are gauge dependent. While performing the conversion of the quark mass to the $\overline{\mathrm{MS}}$ scheme this gauge dependence is compensated by the gauge dependence of the renormalization constant determined in the lattice simulation. The one-loop coefficient $c_m^{(1),x}(\xi)$ of the matching factor $C_m^x = 1 + \frac{\alpha_s}{4\pi}c_m^{(1),x}(\xi)$ as a function of ξ is shown in Fig. 3 for the exceptional and non-exceptional momentum configurations. For almost all gauges the one-loop coefficient of the symmetric subtraction point is smaller than in the asymmetric case except for a small window with

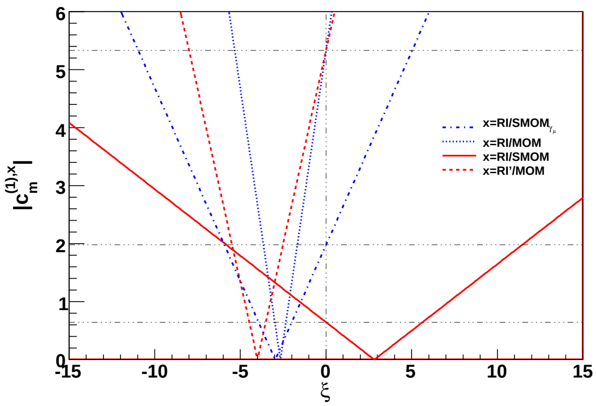

FIG. 3: This figure shows the one-loop coefficients $c_m^{(1),x}$ of the matching factors C_m^x as a function of the gauge parameter ξ for two exceptional (dashed line(red), $x =$ RI$'$/MOM; dotted line(blue), $x =$ RI/MOM) and two non-exceptional (solid line(red), $x =$ RI/SMOM; dashed-dotted line(blue), $x =$ RI/SMOM$_{\gamma_\mu}$) momentum configurations. The horizontal lines indicate the size of the one-loop coefficient in the Landau gauge($\xi = 0$).

$\xi \in (-5.41532..., -3)$ comparing the RI$'$/MOM and RI/SMOM schemes and $\xi \in (-2.74207..., -2.51586...)$ comparing the RI/MOM and RI/SMOM$_{\gamma_\mu}$ schemes.

B. Two-loop anomalous dimensions

The mass anomalous dimension

$$\gamma_m = \frac{d\log m(\mu)}{d\log(\mu^2)} = -\gamma_m^{(0)}\frac{\alpha_s}{\pi} - \gamma_m^{(1)}\left(\frac{\alpha_s}{\pi}\right)^2 - \gamma_m^{(2)}\left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4) \quad (12)$$

can be used to run the quark mass to different energy scales. In the $\overline{\text{MS}}$ scheme the lowest expansion coefficients read

$$\gamma_m^{(0),\overline{\text{MS}}} = \frac{3}{4}C_F, \quad (13)$$

$$\gamma_m^{(1),\overline{\text{MS}}} = \frac{1}{16}\left(\frac{3}{2}C_F^2 + \frac{97}{6}C_F C_A - \frac{10}{3}C_F T_F n_f\right), \quad (14)$$

where $C_F = 4/3 (C_A = 3)$ denotes the Casimir operator in the fundamental(adjoint) representation of SU(3). The symbol n_f represents the number of active fermions and $T_F = 1/2$ is the normalization of the trace of the SU(3) generators in the fundamental representation.

The mass anomalous dimensions of the RI/SMOM and RI/SMOM$_{\gamma_\mu}$ schemes up to two-loop order are given by [14]

$$\gamma_m^{(0),\text{RI/SMOM}} = \gamma_m^{(0),\overline{\text{MS}}} = \gamma_m^{(0),\text{RI/SMOM}_{\gamma_\mu}}, \quad (15)$$

$$\gamma_m^{(1),\text{RI/SMOM}} = \gamma_m^{(1),\overline{\text{MS}}} + \beta^{(0)} C_F 0.121034785..., \quad (16)$$

$$\gamma_m^{(1),\text{RI/SMOM}_{\gamma_\mu}} = \gamma_m^{(1),\overline{\text{MS}}} + \beta^{(0)} C_F 0.371034785..., \quad (17)$$

with the QCD β-function defined by

$$\beta = \frac{d\alpha_s(\mu)/\pi}{d\log(\mu^2)} = -\beta^{(0)}\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3) \quad (18)$$

and the lowest order coefficient

$$\beta^{(0)} = \frac{1}{4}\left(\frac{11}{3}C_A - \frac{4}{3}T_F n_f\right). \quad (19)$$

Similarly one can define the anomalous dimension

$$\gamma_q = 2\frac{d\log\Psi_R}{d\log(\mu^2)}. \quad (20)$$

As shown in Ref. [14] it is equal in the RI/SMOM and RI$'$/MOM schemes and known up to order α_s^3 in Refs. [6, 7]. In the RI/SMOM$_{\gamma_\mu}$ scheme it is given in the Landau gauge by

$$\gamma_q = \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{3}{32}C_F^2 - \frac{31}{192}C_F C_A + \frac{1}{24}C_F T_F n_f\right) + \mathcal{O}(\alpha_s^3) \quad (21)$$

and has been determined in Ref. [14].

IV. CONCLUSIONS

We have discussed the renormalization of quark masses in a regularization-invariant momentum subtraction scheme with a symmetric subtraction point. The scheme is useful in the context of lattice simulations of the light up-, down- and strange-quark masses as an intermediate step before the conversion to the $\overline{\text{MS}}$ scheme. The renormalization constants can be determined by the computation of amputated Green's functions with the insertion of the scalar, pseudoscalar, vector and axial-vector operators. This allows one to derive in continuum perturbation theory the two-loop anomalous dimensions and one-loop matching factors which convert the light quark masses renormalized in the RI/SMOM scheme into the $\overline{\text{MS}}$ scheme. The new RI/SMOM scheme decreases chiral symmetry breaking as well as other unwanted infrared effects in the lattice calculation and the one-loop coefficient of the new matching factor is smaller compared to the one of the RI/MOM(RI$'$/MOM) scheme. A better behaved perturbative series can improve the accuracy of light quark mass determinations, if this behavior is confirmed at higher orders.

Acknowledgments

I would like to thank Y. Aoki, N.H. Christ, T. Izubuchi, C.T Sachrajda and A. Soni for the fruitful collaboration on this subject. This work was supported by U.S. DOE under contract No.DE-AC02-98CH10886.

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[1] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. **B44** (1972) 189.
[2] G. 't Hooft, Nucl. Phys. **B61** (1973) 455.
[3] W. A. Bardeen, A. J. Buras, D. W. Duke and T. Muta, Phys. Rev. **D18** (1978) 3998.
[4] G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa and A. Vladikas, Nucl. Phys. **B445** (1995) 81.
[5] E. Franco and V. Lubicz, Nucl. Phys. **B531** (1998) 641.
[6] K. G. Chetyrkin and A. Retey, Nucl. Phys. **B583** (2000) 3.
[7] J. A. Gracey, Nucl. Phys. **B662** (2003) 247.
[8] W. Celmaster and R. J. Gonsalves, Phys. Rev. **D20** (1979) 1420.
[9] E. Braaten and J. P. Leveille, Phys. Rev. **D24** (1981) 1369.
[10] F. Jegerlehner and O. V. Tarasov, Nucl. Phys. **B549** (1999) 481.
[11] K. G. Chetyrkin and T. Seidensticker, Phys. Lett. **B495** (2000) 74.
[12] K. G. Chetyrkin and A. Retey, hep-ph/0007088.
[13] K. G. Chetyrkin, B. A. Kniehl and M. Steinhauser, Nucl. Phys. **B814** (2009) 231.
[14] C. Sturm et al., Phys. Rev. **D80** (2009) 014501.
[15] C. Allton et al., Phys. Rev. **D78** (2008) 114509.
[16] Y. Aoki et al., Phys. Rev. **D78** (2008) 054510.
[17] S. Weinberg, Phys. Rev. **118** (1960) 838.
[18] K. G. Chetyrkin, J. H. Kühn and A. Kwiatkowski, Phys. Rept. **277** (1996) 189.
[19] T. Blum et al., Phys. Rev. **D66** (2002) 014504.
[20] Y. Aoki, PoS **LATTICE 2008** (2008) 222.
[21] D. J. Broadhurst and A. G. Grozin, Phys. Rev. **D52** (1995) 4082.
[22] J. A. Gracey, Phys. Lett. **B488** (2000) 175.
[23] P. Breitenlohner and D. Maison, Commun. Math. Phys. **52** (1977) 11.
[24] T. L. Trueman, Phys. Lett. **B88** (1979) 331.
[25] S. A. Larin, Phys. Lett. **B303** (1993) 113.
[26] In the Landau gauge at one-loop order the conversion factors C_m are the same for the RI/MOM and RI$'$/MOM schemes.